2012 Su - 2LM ex (Solutions)

Problem 1

1) $u_4 + u_5 = 2u_1 + 2u_2$, so it is $(2,2,0)$.

2) $Su_1 = (0,1,1)$

$$Su_2 = Su_1 + Su_2 - Su_1 = S(u_1+u_2) - Su_1$$
$$= u_4 - (u_2+u_3) = u_1 \quad , \quad (1,0,0)$$
$$Su_3 = u_2 \quad , \quad (0,1,0)$$

$$S \sim \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

3) $\det S \neq 0$ so S is a regular matrix, hence S is bijective

$$Su_3 = u_2 \iff u_3 = S^{-1}(u_2)$$

4) The matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is regular

so u_1, u_2 and u_4 are linearly independent. Since they all belong to $\mathcal{U}$ and $\dim \mathcal{U} = 3$, they span $\mathcal{U}$ and form a basis.

5) We must find α, β, γ such that

$$Su_1 = u_2 + u_3 = \alpha u_1 + \beta u_2 + \gamma u_4 .$$

Then
$$u_2 + u_3 = \alpha u_1 + \beta u_2 + \gamma (u_1 + u_2 + u_3)$$
$$u_2 + u_3 = (\alpha+\gamma) u_1 + (\beta + \gamma) u_2 + \gamma u_3$$

hence $\alpha + \gamma = 0$, $\beta + \gamma = 1$, $\gamma = 1$, so $(\alpha, \beta, \gamma) = (-1, 0, 1)$.

(Or use that $u_3 = u_4 - u_1 - u_2$)

Next

$$Su_2 = u_1 \quad , \text{ that is } (1,0,0)$$

and

$$\begin{aligned} Su_4 &= S(u_1+u_2+u_3) = Su_1 + Su_2 + Su_3 \\ &= -u_1 + u_4 + u_1 + u_2 \\ &= u_2 + u_4 \quad , \text{ that is } (0,1,1) \end{aligned}$$

The matrix is

$$S \sim \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Problem 2

We have that

1)

$$A = QDQ^T \quad \text{where}$$

$$D = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \text{ and } Q = \begin{pmatrix} -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

If we calculate

$$Q\lambda Q^T \text{ with } \lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

we get

$$Q\lambda Q^T = \begin{pmatrix} \frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_3 & -\frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_3 & -\frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_3 \\ -\frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_3 & \frac{1}{6}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{3}\lambda_3 & \frac{1}{6}\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{3}\lambda_3 \\ -\frac{1}{3}\lambda_1 + \frac{1}{3}\lambda_3 & \frac{1}{6}\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{3}\lambda_3 & \frac{1}{6}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{3}\lambda_3 \end{pmatrix}$$

and when $\lambda = D$ we find

$$A = QDQ^T = \begin{pmatrix} 5/3 & 2/3 & 2/3 \\ 2/3 & 13/6 & 1/6 \\ 2/3 & 1/6 & 13/6 \end{pmatrix}$$

2) We have that

$$\ln(A) = Q \ln(D) Q^T, \text{ where}$$

$$\ln(D) = \begin{pmatrix} \ln(1) & & \\ & \ln(2) & \\ & & \ln(3) \end{pmatrix}$$

so from 1)

$$\ln(A) = \begin{pmatrix} \frac{1}{3}\ln(3) & \frac{1}{3}\ln(3) & \frac{1}{3}\ln(3) \\ \frac{1}{3}\ln(3) & \frac{1}{2}\ln(2)+\frac{1}{3}\ln(3) & -\frac{1}{2}\ln(2)+\frac{1}{3}\ln(3) \\ \frac{1}{3}\ln(3) & -\frac{1}{2}\ln(2)+\frac{1}{3}\ln(3) & \frac{1}{2}\ln(2)+\frac{1}{3}\ln(3) \end{pmatrix}$$

3) Since $\det(A) = 1 \cdot 2 \cdot 3 = 6 \neq 0$, A is invertible, and since

$$\det(\ln(A)) = \ln(1)\ln(2)\ln(3) = 0,$$

$\ln(A)$ is <u>not</u>.

4) Since

$$A^{-n} = Q D^{-n} Q^T, \text{ and}$$

$$D^{-n} = \begin{pmatrix} 1^{-n} & & \\ & 2^{-n} & \\ & & 3^{-n} \end{pmatrix},$$ the eigenvalues

are 1, 2^{-n} and 3^{-n}.

Problem 3

1) $\int \sin(x)\sin(2x)\cos(3x)\,dx =$

$$-\frac{1}{8}\int (e^{ix}-e^{-ix})(e^{i2x}-e^{-i2x})(e^{i3x}+e^{-i3x})\,dx =$$

$$-\frac{1}{8}\int (e^{i3x}-e^{-ix}-e^{ix}+e^{-i3x})(e^{i3x}+e^{-i3x})\,dx =$$

$$-\frac{1}{8}\int (e^{i6x}+1-e^{i2x}-e^{-i4x}-e^{i4x}-e^{-i2x}+1+e^{-i6x})\,dx$$

$$-\frac{1}{8}\int (e^{i6x}+e^{-i6x})-(e^{i4x}+e^{-i4x})-(e^{i2x}+e^{-i2x})+2\,dx$$

$$-\frac{1}{4}\int (\cos(6x)-\cos(4x)-\cos(2x)+1)\,dx =$$

$$-\frac{1}{4}\left(\frac{1}{6}\sin(6x)-\frac{1}{4}\sin(4x)-\frac{1}{2}\sin(2x)+x\right)+k =$$

$$\underline{\underline{-\frac{1}{24}\sin(6x)+\frac{1}{16}\sin(4x)+\frac{1}{8}\sin(2x)-\frac{1}{4}x+k.}}$$

2) $6z^2-12z+12=0 \iff z^2-2z+2=0$

$$z=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm i2}{2}=\underline{\underline{1\pm i}}$$

[illegible]

Problem 4

1) $|1-x^2| < 1 \iff -1 < 1-x^2 < 1$, so

$$x \in]-\sqrt{2}, 0[\cup]0, \sqrt{2}[\quad (= A)$$

2) For $\underline{x \in A}$:

$$f(x) = \frac{1}{1-(1-x^2)} = \underline{\underline{\frac{1}{x^2}}}$$

3) $R(f) =]\frac{1}{2}, \infty[$

4) Since, for $\underline{x \in A}$:

$$f'(x) = -2x \sum_{n=0}^{\infty} (n+1)(1-x^2)^n \text{, we have}$$

$$-2x \sum_{n=0}^{\infty} (n+1)(1-x^2)^n = \left(\frac{1}{x^2}\right)' = \frac{-2}{x^3}$$

so $$\sum_{n=0}^{\infty} (n+1)(1-x^2)^4 = \frac{1}{x^4} \quad \text{for } x \in A,$$

hence $$g(x) = \frac{1}{x^4}.$$